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Approximations of Multiattribute Utility Functions*

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1. INTRODUCTION

This paper extends the two-attribute approximation theory for cardinal utility functions in Fishburn [2] to three or more attributes. It is assumed that u is a continuous real valued utility function on a closed and bounded rectangular subset T of n-dimensional Euclidean space and that u is unique up to positive affine transformations of the form u^{ab} where $u^{ab}(x) = au(x) + b$ with $a > 0$. For expositional simplicity we shall let $T = [0, 1]^n$.

Each approximation v for u on T that is discussed is a simple algebraic combination of univariate functions and is of the form

$$v(x_1, \ldots, x_n) = \sum_{j=1}^{k} f_{1j}(x_1) f_{2j}(x_2) \cdots f_{nj}(x_n). \tag{1}$$

The distance between u and v that we shall use is the uniform norm $D(v, u) = \sup | v(x) - u(x)|$. Because of the added complexities of higher dimensions, only simple approximations of form (1) will be examined. The next two sections consider, respectively, the simple additive and multiplicative approximations. The final section then briefly looks at three other approximations. All but the last approximation use n or more univariate conditional utility functions. The last approximation is a multilinear interpolation form that only requires estimation of u at the 2^n vertices of T.

As in Fishburn [2] we shall say that v is *affine preserving* if and only if, for all $a > 0$ and b, $v^{ab}(x) = av(x) + b$ is equal to the right side of (1) for all $x \in T$ when every occurrence of u on the right side of (1) is replaced by $au + b$. We shall let $v_{ab}(x)$ denote the right side of (1) when u therein is replaced by $au + b$. Hence v is affine preserving when $v_{ab}(x) = v^{ab}(x)$ for all $a > 0$, b and $x \in T$.

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Approximation v is *monotonicity preserving* in x_i if and only if v is monotonic increasing (decreasing) in x_i whenever u is monotonic increasing (decreasing) in x_i. And v is *monotonicity preserving* if it is monotonicity preserving in all n variables.

The utility function u will be said to be *conservative* if and only if it strictly increases in all n variables and $u(x) + u(y) > u(z) + u(w)$ whenever $x, y, z, w \in T$ and there are distinct $i, j \in \{1,..., n\}$ such that $x_i = z_i > y_i = w_i$, $y_j = z_j > x_j = w_j$, and $x_k = y_k = z_k = w_k$ for all $k \notin \{i, j\}$. This definition corresponds to Richard's [4] conception of strict multivariate risk aversion. Approximation v is *conservatism preserving* if and only if v is conservative whenever u is conservative.

2. Additive Approximations

The basic results for the simple additive approximation that uses one conditional utility function for each attribute are given in our first theorem. Refinements for the additive approximation are discussed later in the section. Here and later we shall let $u_0(x_i) = u(x_1^0,..., x_{i-1}^0, x_i, x_{i+1}^0,..., x_n^0)$ when x^0 is a fixed point in T. Although this notation is ambiguous in the sense that $u_0(.5)$ does not tell which i is referred to, it is typographically simple and should cause no confusion.

Theorem 1. *Given fixed* $x^0 = (x_1^0,..., x_n^0) \in T$ *let*

$$v(x) = \sum_{i=1}^{n} u_0(x_i) - (n - 1)\, u(x^0) \qquad \text{for all} \quad x \in T. \tag{2}$$

Then $v(x) = u(x)$ whenever $x_i = x_i^0$ for at least $n - 1$ of the $i \in \{1,..., n\}$, and v is affine preserving and monotonicity preserving. In addition, let $M = \max_T u(x)$, $m = \min_T u(x)$ and $W = M - m$. If x^0 is chosen so that $u(x^0) = (M + m)/2$ then, when $n \geqslant 2$,

(a) *$D(v, u) \leqslant W(n + 1)/2$ if u is not monotonic in more than $n - 2$ variables;*

(b) *$D(v, u) \leqslant Wn/2$ if u is monotonic in $n - 1$ variables;*

(c) *$D(v, u) \leqslant W(n - 1)/2$ if u is monotonic in all n variables.*

Remarks. This theorem subsumes Theorem 1 in [2] for $n = 2$. It is clearly not very encouraging for the additive approximation for larger n since, for example, $D(v, u)$ could well exceed $W = \max u(x) - \min u(x)$ even when u is monotonically increasing in every variable and x^0 is a point that has the mid-value $u(x^0) = (M + m)/2$, provided that $n \geqslant 4$. Note also that monotonicity does not affect the upper bound on $D(v, u)$ unless it holds for at least $n - 1$ variables.

Proof. The first part of Theorem 1 is obvious from (2). The latter part, with M, m, W and x^0 given, can be proved by worst-case arguments. By change of variables if necessary, it will suffice to consider u monotonically increasing in its first k variables for $k \in \{0, n-2, n-1, n\}$. For $k = 0$, a worst case is $u_0(x_i) = M$ for all i and $u(x) = m$, in which case $D(v, u) = nM - (n-1)(M+m)/2 - m = W(n+1)/2$. For $k = n-2$ with $x_i^0 < 1$ for each $i \leqslant n-2$, we could have each $u_0(x_i)$ very near to M for all i with $u(x)$ near to m, so again $D(v, u) \leqslant W(n+1)/2$. A worst case for $k = n-1$ has $x_i > x_i^0$ for $i = 1,..., n-2$ and $x_{n-1} < x_{n-1}^0$ with $u_0(x_i)$ near to M for all $i \leqslant n-2$, $u_0(x_{n-1})$ slightly less than $(M+m)/2$ and $u_0(x_n) - u(x)$ near to $M - m$. (If we take $x_i > x_i^0$ for all $i \leqslant n-1$, then $u_0(x_n) - u(x)$ must be negative, but with $x_i - x_i^0$ of different signs for different $i \leqslant n-1$ the sign of $u_0(x_n) - u(x)$ is not determined.) So for $k = n-1$ we get $D(v, u) \leqslant (n-2)M + (M+m)/2 - (n-1)(M+m)/2 + (M-m) = Wn/2$. Finally, for $k = n$ a worst case is $u(x)$ and all $u_i(x_i^0)$ near to M, hence $D(v, u) \leqslant nM - (n-1)(m+M)/2 - M = W(n-1)/2$. Q.E.D.

If utilities are fully additive over the attributes then $D(v, u) = 0$ when (2) is used. More generally, if the attributes can be grouped into subsets such that utilities are additive among the subsets, then Theorem 1 can be used for each subset with two or more attributes. Suppose for example that $\{I_1, ..., I_N\}$ is a partition of $\{1,..., n\}$ with $|I_j| = n_j \geqslant 1$ for $j = 1,..., N$ such that there is a real valued function u_j on $[0, 1]^{n_j}$ for each j with

$$u(x) = \sum_{j=1}^{N} u_j(x(I_j)) \qquad \text{for all} \quad x \in T. \tag{3}$$

where $x(I_j)$ is the n_j-tuple of x_i for $i \in I_j$. Let $M_j = \max u_j(x(I_j))$, $m_j = \min u_j(x(I_j))$ and $W_j = M_j - m_j$ for each j—so that $M = \sum M_j$, $m = \sum m_j$ and $W = \sum W_j$ in Theorem 1—and let x^0 satisfy $u_j(x^0(I_j)) = (M_j + m_j)/2$ for each j. Then, when v_j is an additive approximation of u_j like (2), $D(v_j, u_j) = 0$ if $n_j = 1$ and, for $n_j > 1$, $D(v_j, u_j)$ is bounded above by $W_j(n_j+1)/2$, $W_j n_j/2$ or $W_j(n_j - 1)/2$ according to whether u_j is monotonic in fewer than $n_j - 1$, exactly $n_j - 1$, or n_j variables. In addition, with $v = v_1 + \cdots + v_N$, it follows that

$$D(v, u) \leqslant \sum_{j=1}^{N} D(v_j, u_j).$$

Hence if u is monotonic in all variables then $D(v, u) \leqslant \sum_j W_j(n_j - 1)/2 \leqslant (\max W_j)(n - N)/2$.

The preceding paragraph shows that more information about u than is presumed in Theorem 1 allows tighter bounds on $D(v, u)$. A similar procedure allows the following refinement without assuming partial additivity as in (3). This refinement can of course be used in connection with (3) when (3) holds.

THEOREM 2. *Suppose* (2) *holds with* M, m, W *and* x^0 *as given in Theorem* 1 *and suppose further that* $M_i = \max u_0(x_i)$, $m_i = \min u_0(x_i)$ *for* $i = 1,\ldots, n$ *and that* u *is monotonically increasing in its first* k *variables. Then, when* $u_0(x_i^0) = u(x^0) = (M+m)/2$ *for all* i, *and* $n \geqslant 2$:

$$\text{(a)}\quad D(v, u) \leqslant \max\left\{\sum_{i=1}^{n}\left(M_i - \frac{M+m}{2}\right), \sum_{i=1}^{n}\left(\frac{M+m}{2} - m_i\right)\right\} + W/2 \qquad \text{if } k \leqslant n-2;$$

$$\text{(b)}\quad D(v, u) \leqslant \max\left\{\sum_{i=1}^{n-1}\left(M_i - \frac{M+m}{2}\right) + M_n - \min_{i\leqslant n-1} M_i, \sum_{i=1}^{n-1}\left(\frac{M+m}{2} - m_i\right) - m_n + \max_{i\leqslant n-1} m_i\right\} + W/2 \qquad \text{if } k = n-1;$$

$$\text{(c)}\quad D(v, u) \leqslant \max\left\{\sum_{i=1}^{n}\left(M_i - \frac{M+m}{2}\right) + M + m - \min\{M_i + M_j\colon 1 \leqslant i < j \leqslant n\}, \sum_{i=1}^{n}\left(\frac{M+m}{2} - m_i\right) - M - m + \max\{m_i + m_j\colon 1 \leqslant i < j \leqslant n\}\right\} + W/2 \qquad \text{if } k = n.$$

Proof. In each of (a), (b), and (c) the M_i part comes from a worst-case maximization of $v(x) - u(x)$, and the m_i part comes from a worst-case maximization of $u(x) - v(x)$. I shall prove only the M_i parts of (b) and (c) since their m_i proofs are symmetric and since (a) is obvious. For $k = n - 1$ in (b), $v(x) - u(x) = \sum_{i=1}^{n-1} [u_0(x_i) - (M+m)/2] + [u_0(x_n) - u(x)]$. If we allow one $x_i < x_i^0$ for $i < n$ then $u_0(x_n) - u(x)$ can be made near to $M_n - m$ and the $u_0(x_j)$ for $j \neq i$ and $j \leqslant n-1$ can be taken near to their M_j. We also choose the i for $x_i < x_i^0$ as the i with the smallest M_i and make $u_0(x_i)$ slightly less than $u(x^0)$. It is easily checked that this "tends" to maximize $v(x) - u(x)$ and it implies that $v(x) - u(x) \leqslant M_n - m + \sum_{i\leqslant n-1}(M_i - (M+m)/2) - \min\{M_i\colon i \leqslant n-1\} + (M+m)/2 = \sum_{i\leqslant n-1}(M_i - (M+m)/2) + M_n - \min\{M_i\colon i \leqslant n-1\} + W/2$. Finally, for $k = n$ in (c), the max of

$v(x) - u(x)$ will occur with all $u_0(x_i) = M_i$ except for 0, 1 or 2 i for which we take $u_0(x_i)$ slightly less than $u(x^0)$. The worst case here arises if we choose two i for $u_0(x_i) < u(x^0)$, in which case $u(x)$ can be as small as m. When the two i are chosen so that their M_i are as small as possible, the result is $v(x) - u(x) \leqslant \sum_{i=1}^n (M_i - (M+m)/2) + (M+m)/2 - \min\{M_i + M_j : 1 < i < j < n\} + 2[(M+m)/2] - m = \sum_{i=1}^n (M_i - (M+m)/2) + M + m - \min\{M_i + M_j\} + W/2$. Q.E.D.

Fishburn [2] shows that if (2) is used when u is conservative and $n = 2$, and if $\Delta = u(1, 0) + u(0, 1) - u(0, 0) - u(1, 1)$, then $D(v, u)$ cannot be less than $\Delta/4$ but x^0 can be chosen for (2) to ensure that $D(v, u) < \Delta/3$. Because $\Delta \leqslant W$, the $\Delta/3$ bound is less than the upper bound in Theorem 2(c), which is never less than $W/2$. Although the conservatism picture is less clear when $n \geqslant 3$, several results can be established for this case. We begin with two lemmas.

LEMMA 1. *Suppose u is conservative, $x_i \leqslant y_i$ for $i = 1,..., n$, and I and J are nonempty disjoint subsets of $\{1,..., n\}$ with $I \cup J = \{1,..., n\}$. Then*

$$u(x_i \text{ for } i \in I, y_j \text{ for } j \in J) + u(y_i \text{ for } i \in I, x_j \text{ for } j \in J) \geqslant u(x) + u(y), \tag{4}$$

$$\sum_{i=1}^{n} u(x_i, y_j \text{ for all } j \neq i) \geqslant u(x) + (n-1)\, u(y), \tag{5}$$

$$\sum_{i=1}^{n} u(y_i, x_j \text{ for all } j \neq i) \geqslant u(y) + (n-1)\, u(x), \tag{6}$$

and $>$ holds for each of (4), (5) *and* (6) *if $x_i < y_i$ for some i.*

LEMMA 2. *Suppose u is conservative, v is given by* (2), *and $j \in \{1,..., n\}$. If $x_i \leqslant x_i^0$ for all $i \neq j$, then $v(x) - u(x)$ strictly decreases in x_j when the x_i for $i \neq j$ are fixed; if $x_i \geqslant x_i^0$ for all $i \neq j$, then $v(x) - u(x)$ strictly increases in x_j when the x_i for $i \neq j$ are fixed.*

Proof of Lemma 1. Let u be conservative with $x_i \leqslant y_i$ for $i = 1,..., n$. If $I = \{1\}$ then conservatism implies

$$\begin{aligned} &u(x_1, y_2, ..., y_j, x_{j+1}, ..., x_n) + u(y_1, ..., y_{j-1}, x_j, ..., x_n) \\ &\qquad \geqslant u(y_1, ..., y_j, x_{j+1}, ..., x_n) + u(x_1, y_2, ..., y_{j-1}, x_j, ..., x_n) \end{aligned}$$

for $j = 2,..., n$. Addition of these inequalities over j from 2 to n, plus cancellation of identical terms, yields (4) for $I = \{1\}$. Since the same procedure holds for any $I = \{i\}$, (4) holds when $|I| = 1$. Proceeding by induction, suppose (4) holds for $|I| = k - 1 \geqslant 1$. This hypothesis and the result just proved for $|I| = 1$ respectively imply $u(x_1, ..., x_k, y_{k+1}, ..., y_n) + u(x_1, y_2, ..., y_k, x_{k+1}, ..., x_n) \geqslant u(x) + u(x_1, y_2, ..., y_n)$ and $u(x_1, y_2, ..., y_n) + u(y_1, ..., y_k, x_{k+1}, ..., x_n) \geqslant u(x_1, y_2, ..., y_k, x_{k+1}, ..., x_n) + u(y)$, the sum of which yields (4) for $I = \{1, ..., k\}$. It follows that (4) holds in general. Using (4) we then have

$$u(x_1, y_2, ..., y_n) + u(y_1, x_2, y_3, ..., y_n) \geqslant u(x_1, x_2, y_3, ..., y_n) + u(y),$$
$$u(x_1, ..., x_k, y_{k+1}, ..., y_n) + (y_1, ..., y_k, x_{k+1}, y_{k+2}, ..., y_n)$$
$$\geqslant u(x_1, ..., x_{k+1}, y_{k+2}, ..., y_n) + u(y) \qquad \text{for} \quad k = 2, ..., n-1,$$

and the addition of these $n - 1$ inequalities implies (5). Inequality (6) is proved in a similar way. If $x_i < y_i$ for some i then it follows from the procedures used to establish (4), (5) and (6) that they will hold with $\geqslant$ replaced by $>$. Q.E.D.

Proof of Lemma 2. Given the hypotheses of the lemma suppose for definiteness that $j = 1$. Then $v(x) - u(x) = \sum_{i>1}[u_0(x_i) - u(x^0)] + [u_0(x_1) - u(x)]$. If the x_i for $i > 1$ are fixed at values $x_i \leqslant x_i^0$, and if $x_i < y_1$, then (4) with $I = \{1\}$ implies that $u_0(x_1) - u(x_1, x_2, ..., x_n) > u_0(y_1) - u(y_1, x_2, ..., x_n)$. Therefore $v(x) - u(x)$ decreases as x_1 increases with the $x_i \leqslant x_i^0$ fixed for $i > 1$. The proof of the final part of Lemma 2 is similar. Q.E.D.

Using Lemmas 1 and 2, we now consider what happens to $v(x) - u(x)$ and $u(x) - v(x)$ when $n \geqslant 3$, u is conservative, and v is given by (2) with $u(x^0) = (M + m)/2$. Suppose first that the maximum of $u(x) - v(x)$ occurs at x for which $x_i < x_i^0$ for $i \in I$ and $x_i \geqslant x_i^0$ for $i \in J$. If x^0 is in the interior of T then Lemma 2 implies that neither I nor J is empty, and if x^0 is not in the interior then the definition of I and J can be modified if necessary ($\leqslant$ for I, $>$ for J) to ensure that neither I nor J is empty. Then, by (5) and (6) respectively,

$$\sum_I u_0(x_i) \geqslant u(x_i \text{ on } I, x_i^0 \text{ on } J) + (|I| - 1)\, u(x^0),$$

$$\sum_J u_0(x_i) \geqslant u(x_i^0 \text{ on } I, x_i \text{ on } J) + (|J| - 1)\, u(x^0),$$

so that $\sum u_0(x_i) \geqslant u(x_i \text{ on } I, x_i^0 \text{ on } J) + u(x_i^0 \text{ on } I, x_i \text{ on } J) + (n-2)u(x^0)$. It then follows from (2) that

$$u(x) - v(x) \leqslant u(x) + u(x^0) - u(x_i \text{ on } I, x_i^0 \text{ on } J) - u(x_i^0 \text{ on } I, x_i \text{ on } J) \leqslant W/2.$$

Therefore, when u is conservative and $u(x^0) = (M + m)/2$, $u(x) - v(x)$ cannot exceed $W/2$.

Consider next the maximization of $v(x) - u(x)$ when u is conservative, and let $M_i = \max\ u_0(x_i)$ and $m_i = \min\ u_0(x_i)$. For convenience we examine $v(x) - u(x)$ when $x_i \leqslant x_i^0$ for $i = 1,\dots, k$ and $x_i \geqslant x_i^0$ for $i > k$. When $k = 0$, Lemma 2 implies that $v(x) - u(x)$ is maximized at $x = (1,\dots, 1)$, where

$$v(1,\dots, 1) - u(1,\dots, 1) = \sum_{i=1}^{n} \left(M_i - \frac{M + m}{2}\right) - W/2,$$

assuming that $u(x^0) = (M + m)/2$. Similarly, when $k = n$, Lemma 2 implies that $v(x) - u(x)$ is maximized at $x = (0,\dots, 0)$, where

$$v(0,\dots, 0) - u(0,\dots, 0) = W/2 - \sum_{i=1}^{n} \left(\frac{M + m}{2} - m_i\right).$$

Inequalities (6) and (5) imply respectively that $v(1,\dots, 1) - u(1,\dots, 1) \geqslant 0$ and $v(0,\dots, 0) - u(0,\dots, 0) \geqslant 0$. Since Lemma 2 implies that $v(x) - u(x)$ cannot exceed $v(1,\dots, 1) - u(1,\dots, 1)$ when $k = 1$ and that it cannot exceed $v(0,\dots, 0) - u(0,\dots, 0)$ when $k = n - 1$, it remains only to examine $k \in \{2,\dots, n - 2\}$ when $n \geqslant 4$. In the latter case a worst-case argument shows that

$$v(x) - u(x) \leqslant \sum_{i=k+1}^{n} \left(M_i - \frac{M + m}{2}\right) + W/2,$$

and the worst of these worst cases occurs when $k = 2$. Since it is easily seen that the upper bound in the preceding expression with $k = 2$ exceeds both $v(1,\dots, 1) - u(1,\dots, 1)$ and $v(0,\dots, 0) - u(0,\dots, 0)$, and since the two i for which $x_i \leqslant x_i^0$ could be any two of the $i \in \{1,\dots, n\}$, we have established the fact that $v(x) - u(x)$ is bounded by $\sum_{i=1}^{n}(M_i - (M + m)/2) + M + m - \min\{M_i + M_j\colon 1 \leqslant i < j \leqslant n\} + W/2$.

The following theorem summarizes the foregoing conclusions.

THEOREM 3. *Suppose $n \geqslant 3$, u is conservative and v is given by* (2) *with* $u(x^0) = (M + m)/2$, $M = \max\ u(x)$, $m = \min\ u(x)$, $W = M - m$, *and* $M_i = \max u_0(x_i)$, $m_i = \min u_0(x_i)$ *for $i = 1,\dots, n$. Then* $\max[u(x) - v(x)] \leqslant W/2$, *and*

$$n = 3 \Rightarrow \max_T\ [v(x) - u(x)]$$
$$= \max \left\{\sum_{i=1}^{3} \left(M_i - \frac{M + m}{2}\right) - W/2,\ W/2 - \sum_{i=1}^{3} \left(\frac{M + m}{2} - m_i\right)\right\},$$

$$n \geqslant 4 \Rightarrow \max_T\ [v(x) - u(x)]$$
$$\leqslant \sum_{i=1}^{n} \left(M_i - \frac{M + m}{2}\right) + M + m$$
$$- \min\{M_i + M_j\colon 1 \leqslant i < j \leqslant n\} + W/2.$$

Although the bound on $v(x) - u(x)$ for $n \geqslant 4$ may be no better than the bound on $D(v, u)$ in Theorem 2(c), other choices of $u(x^0)$ under conservatism may give better general bounds. For example, if u is conservative and if $x^0 = (1,..., 1)$, then Lemma 2 shows that $D(v, u) = v(0,..., 0) - u(0,..., 0) = u(0, 1,..., 1) + \cdots + u(1,..., 1, 0) - (n-1)u(1,..., 1) - u(0,..., 0)$, which is strictly positive by (5) but can never exceed W.

3. Multiplicative Approximations

The basic multiplicative approximation for u with fixed point x^0 and $u_0 \equiv u(x^0) \neq 0$ is

$$v(x) = \prod_{i=1}^{n} u_0(x_i)/u_0^{n-1} \qquad \text{for all} \quad x \in T. \tag{7}$$

This is exact when $x_i = x_i{}^0$ for at least $n-1$ of the $i \in \{1,..., n\}$, it is monotonicity preserving if u has constant sign, and it is not generally affine preserving. When $au_0 + b \neq 0$, the affine transformation $u^{ab} = au + b$ on the right side of (7) gives $v_{ab}(x) = \prod (au_0(x_i) + b)/(au_0 + b)^{n-1}$ with $v_{ab}(x) = v^{ab}(x) = av(x) + b$ if at least $n-1$ of the i have $x_i = x_i{}^0$. When the v_{ab} are normalized by the transformations $w_{ab}(x) = (v_{ab}(x) - b)/a$, we get

$$w_{ab}(x) = \frac{\prod (au_0(x_i) + b) - b(au_0 + b)^{n-1}}{a(au_0 + b)^{n-1}}. \tag{8}$$

The family $\{w_{ab}\colon a > 0,\ au_0 + b \neq 0\}$ is the set of basic multiplicative approximations for u with fixed point x^0. The different functions in this family correspond to different choices of origin and scale unit for u.

Because a family $\{w_{ab}\}$ of multiplicative approximations corresponds to each fixed point x^0, multiplicative approximations are more flexible than additive approximations. An example of this flexibility is shown by the fact that any additive approximation can be approximated to any desired accuracy by a multiplicative approximation. This is shown by the next theorem. On the other hand, a multiplicative approximation cannot generally be approximated to any desired accuracy by an additive approximation, as can be seen by supposing that $u(x) = x_1x_2$ on $[0, 1]^2$ for $n = 2$. Then $D(v, u) = 0$ when v is given by (7) with $x^0 = (1, 1)$; but, when v is given by (2), $D(v, u)$ is minimized at $x^0 = (1/2, 1/2)$, where its value is $1/4$.

Theorem 4. *Suppose x^0 is the fixed point for* (2) *and* (7), *$u(x^0) \neq 0$, $b \neq 0$, and v is given by* (2). *Then for every $\delta > 0$ there is an $a > 0$ for which $D(w_{ab}, v) < \delta$.*

Proof. The b^n terms in the numerator of (8) cancel and we are left with $ab^{n-1}v(x)$ plus terms in a^2 through a^n. When the leading a in the denominator of (8) cancels into the numerator we are left with

$$w_{ab}(x) = \frac{b^{n-1}v(x) + \text{terms in } a \text{ through } a^{n-1}}{(au_0 + b)^{n-1}},$$

where $v(x)$ is given by (2). With $a > 0$ and small, and $b \neq 0$, it follows that $w_{ab}(x) \to v(x)$ as $a \to 0$, and the convergence of w_{ab} to v is easily seen to be uniform. Q.E.D.

The next theorem, which corresponds to Theorem 5 in [2], shows how much v might differ from u when v is given by (7). The theorem considers all cases in which $\max u(x) - \min u(x) = 1$ with $\min u(x) \geqslant -1/2$. A scale transformation that maps u into au, $a > 0$, will map $D(v, u)$ into $aD(v, u)$.

THEOREM 5. *Suppose $n \geqslant 3$ and v is given by* (7) *with* $\min u = r$, $\max u = r + 1$ *and $u_0 = u(x^0) \neq 0$. If $-1/2 \leqslant r < 0$ then it is always possible to have $D(v, u) \leqslant 1$ by choosing $u_0 = r + 1$. If $r \geqslant 0$ then*:

(a) *If u is monotonic in no more than $n - 2$ variables, it is always possible to have $D(v, u) \leqslant [(r + 1)^{n+1} - r^{n+1}]/[(r + 1)^n + r^n]$ by choosing $u_0^{n-1} = [(r + 1)^n + r^n]/(2r + 1)$;*

(b) *If u is monotonic in $n - 1$ variables, it is always possible to have $D(v, u) \leqslant [(r + 1)^n - r^n]/[(r + 1)^{n-1} + r^{n-1}]$ by choosing $u_0^{n-2} = [(r + 1)^{n-1} + r^{n-1}]/(2r + 1)$;*

(c) *If u is monotonic in all variables and $n = 3$, it is always possible to have $D(v, u) \leqslant (2r^2 + 3r + 1)/(2r^2 + 2r + 1)$ by choosing $u_0^2 = r^2 + r + 1/2$;*

(d) *If u is monotonic in all variables and $n \geqslant 4$, then $D(v, u) \leqslant [u_0^2(r + 1)^{n-2} - r^n]/[u_0^2(r + 1)^{n-3} + r^{n-1}]$ when u_0 is the positive real root of*

$$u_0^{n-1}(2r + 1) - u_0^2(r + 1)^{n-2} = r^{n-1}(r + 1), \tag{9}$$

and there is no other value of u_0 that can guarantee a smaller upper bound on $D(v, u)$.

Remarks. The bounds on $D(v, u)$ given prior to part (d) are also the best possible without assuming more about u. Monotonicity has no effect on the upper bound when the origin is interior to $u(T)$, but is important when $\min u(x) \geqslant 0$. In each of (a) through (d), $D(v, u) \leqslant 1$ when $r = 0$; as $r \to \infty$ the bounds on $D(v, u)$ in (a) through (d) respectively approach $(n + 1)/2$, $n/2$, $(n - 1)/2$ and $(n - 1)/2$, which are the same as the respective bounds in Theorem 1 when $W = 1$. Hence for larger n the upper bounds on $D(v, u)$ with $r = 0$ in the multiplicative approximation are considerably better than the general bounds for the additive approximation. As will be shown in the following proof, there is an important difference between the $n = 3$ and $n \geqslant 4$ cases when $r \geqslant 0$ and u is monotonic in all variables. It may also be noted that when $r = 0$ in part (c), any $u_0^2 \in [1/2, 1]$ will give $D(v, u) \leqslant 1$. Hence in all cases we can guarantee that v is monotonicity preserving and that $D(v, u) \leqslant 1$ by taking $r = 0$ and $u_0 = 1$.

Proof. Throughout this proof E is an abbreviation for $|v(x) - u(x)| = |\prod u_0(x_i)/u_0^{n-1} - u(x)|$ and u is assumed to increase in x_i if it is monotonic in x_i. (If u decreases in x_i, a change of variable from x_i to $1 - x_i$ gives the same conclusions.)

Given the hypotheses of Theorem 5, assume throughout this paragraph that $-1/2 \leqslant r < 0$. Suppose first that $u_0 > 0$. With no monotonicity, $\max\ E \leqslant \max\{(r+1)^n/u_0^{n-1} - r,\ (r+1) - r(r+1)^{n-1}/u_0^{n-1}\}$. The latter max is minimized at $u_0 = r + 1$, where $\max E \leqslant 1$. Even if u is monotonic in every variable, by taking $x_i < x_i^0$ for one i it is still possible to have a worst-case value of $u(x) - v(x)$ near to $(r+1) - r(r+1)^{n-1}/u_0^{n-1}$, which is minimized at $u_0 = r + 1$ with value 1. Suppose next that $u_0 < 0$. If n is odd then $u_0^{n-1} > 0$ and we cannot improve on the $u_0 > 0$ result since $r + 1 \geqslant |r|$. If n is even then E could be as large as $(r+1) - (r+1)^n/u_0^{n-1}$, which is as great as 1 when $r \leqslant u_0 < 0$. Hence, when $r < 0$, we cannot improve on $\max E \leqslant 1$ at $u_0 = r + 1$.

We assume henceforth in this proof that $r \geqslant 0$. If u is monotonic in no more than $n - 2$ variables, worst-case considerations give $\max E \leqslant \max\{(r+1)^n/u_0^{n-1} - r,\ (r+1) - r^n/u_0^{n-1}\}$. The latter max is minimized when its terms are equal, i.e. when $u_0^{n-1} = [(r+1)^n + r^n]/(2r+1)$. This value of u_0 is in $[r, r+1]$, and it implies that $\max E \leqslant [(r+1)^{n+1} - r^{n+1}]/[(r+1)^n + r^n]$.

Suppose next that u is monotonic in its first $n - 1$ variables. The worst-case values of $v(x) - u(x)$ will arise either with all $x_i \geqslant x_i^0$ for $i = 1, \ldots, n-1$ or with $x_i \geqslant x_i^0$ for $i \leqslant n-2$ and $x_{n-1} < x_{n-1}^0$, and hence $\max[v(x) - u(x)] \leqslant \max\{(r+1)^n/u_0^{n-1} - (r+1),\ (r+1)^{n-1}/u_0^{n-2} - r\}$. The worst-case value of $u(x) - v(x)$ is obtained either with $x_i \leqslant x_i^0$ for $i = 1, \ldots, n-1$ or with $x_i \leqslant x_i^0$ for $i \leqslant n-2$ and $x_{n-1} > x_{n-1}^0$, and therefore $\max[u(x) - v(x)] \leqslant \max\{(r+1) - (r+1)\, r^{n-1}/u_0^{n-1},\ (r+1) - r^{n-1}/u_0^{n-2}\}$. Since the second term in the latter max is never less than the first term,

$$\max E \leqslant \max\{(r+1)^n/u_0^{n-1} - r - 1,$$
$$(r+1)^{n-1}/u_0^{n-2} - r,\ r + 1 - r^{n-1}/u_0^{n-2}\}.$$

The first two terms on the right side of this inequality decrease in u_0 and the third term increases in u_0. The second and third terms are equal when $u_0^{n-2} = [(r+1)^{n-1} + r^{n-1}]/(2r+1)$ with value $[(r+1)^n - r^n]/[(r+1)^{n-1} +$

$r^{n-1}]$, which is the minimum of the right side if this value is as great as the first term's value at the indicated u_0. Thus to complete the proof of part (b) of the theorem we need to show that

$$(r+1)^n \Big/ \left[\frac{r^{n-1}+(r+1)^{n-1}}{2r+1}\right]^{(n-1)/(n-2)} - r - 1 \leqslant \frac{(r+1)^n - r^n}{(r+1)^{n-1}+r^{n-1}}.$$

After some algebraic manipulation, this inequality can be written as

$$(2r+1)[(2r+1)(r+1)^n]^{n-2} \leqslant [(r+1)^{n-1}+r^{n-1}][2(r+1)^n + r^{n-1}]^{n-2}.$$

This is true since $(2r+1)^{n-1}(r+1)^{n(n-2)} \leqslant 2^{n-2}(r+1)^{n(n-2)}[(r+1)^{n-1} + r^{n-1}]$, or $(2r+1)^{n-1} \leqslant 2^{n-2}[(r+1)^{n-1} + r^{n-1}]$, as the reader can readily show.

Finally, suppose that u is monotonic in all n variables. For worst cases we consider $x_i \geqslant x_i^0$ for n, $n-1$ or $n-2$ variables for $v-u$, and $x_i \leqslant x_i^0$ for n, $n-1$, or $n-2$ variables for $u-v$. In the $v-u$ case, the worst $n-1$ case is dominated by the worst n case; for $u-v$, the worst n case is dominated by the worst $n-1$ case, as is easily proved. This leaves us with

$$\max E \leqslant \max\{(r+1)^n/u_0^{n-1} - r - 1,\ (r+1)^{n-2}/u_0^{n-3} - r,$$
$$r+1-(r+1)\,r^{n-1}/u_0^{n-1},\ r+1-r^{n-2}/u_0^{n-3}\}.$$

When $n=3$, the second and fourth terms on the right side equal 1, and the first and third equal $(2r^2+3r+1)/(2r^2+2r+1)$ when $u_0^2 = r^2 + r + 1/2$. This verifies part (c) of the theorem. Part (d) is clearly true when $r=0$.

Assume henceforth that $n \geqslant 4$, $r > 0$ and u is monotonic in all variables. It is easily seen that the right side of the preceding max E inequality is minimized when one of its first terms equals one of its last two terms. We shall prove that the minimum occurs when the second and third terms are equal, i.e. when $(r+1)^{n-2}/u_0^{n-3} - r = (r+1) - (r+1)\,r^{n-1}/u_0^{n-1}$. The applicable value of u_0 that satisfies this equation and the corresponding bound of $D(v, u)$ are given in part (d) of the theorem. To complete the proof we need to show that, when u_0 is the positive root of (9), the first and fourth terms on the right of the preceding max E inequality cannot be greater than the second or third term. Because the first two terms decrease in u_0 and the last two increase in u_0, it will suffice to show that the value of u_0 at which the first and third terms are equal is less than u_0 given by (9), and that the value of u_0 at which the second and fourth are equal is greater than u_0 by (9).

Consider the fourth term, i.e. $r+1-r^{n-2}/u_0^{n-3}$. This equals the second term iff $u_0^{n-3} = [(r+1)^{n-2} + r^{n-2}]/(2r+1)$. At this value of u_0 the third term exceeds the fourth term iff $u_0^2 > r(r+1)$, or, after substitution and simplification, iff $[(r+1)^{(n-1)/2} - r^{(n-1)/2}][(r+1)^{(n-3)/2} - r^{(n-3)/2}] > 0$, which is obviously true. Since the third term exceeds the fourth when the second and fourth are equal, and since the second decreases in u_0 while the third and fourth increase in u_0, the value of u_0 at which the second and third are equal must be less than the value of u_0 at which the second and fourth are equal.

We now examine the first term, i.e. $(r+1)^n/u_0^{n-1} - (r+1)$. This equals the third term iff $u_0^{n-1} = [(r+1)^{n-1} + r^{n-1}]/2$. At this value of u_0 the second term exceeds the first term iff $(r+1)^{n-2}/u_0^{n-3} - r > (r+1)^n/u_0^{n-1} - (r+1)$ which, after substitution and algebraic manipulation, occurs iff

$$2^{n-3} > \frac{[(r+1)^{n-1}(2r+1) - r^{n-1}]^{n-1}}{(r+1)^{(n-1)(n-2)}\,[(r+1)^{n-1} + r^{n-1}]^2}.$$

The right hand side of this inequality equals 1 at $r = 0$ and approaches 2^{n-3} as $r \to \infty$. (The latter value is most easily shown by expanding numerator and denominator in powers of r. The numerator equals $2^{n-1}r^{2n-2}$ plus terms in smaller powers of r, and the denominator equals $4r^{2n-2}$ plus terms in smaller powers of r.) Moreover, it can also be shown that the derivative of the right side with respect to r is positive. Since my proof of this is long but algebraically straightforward, I shall not present it here. It then follows that the preceding inequality is true for all $r \geqslant 0$. Hence the second term exceeds the first term when the first and third are equal. The monotonicity aspects of the terms then allow us to conclude that the u_0 value at which the first and third terms are equal is less than the u_0 value at which the second and third terms are equal. Q.E.D.

Equation (3) of the preceding section expresses a case in which the attributes can be grouped into subsets such that utilities are additive among the subsets. Given (3), one could approximate each u_j in (3) by a simple multiplicative rather than additive approximation. For example, if (3) holds and u and the u_j are scaled so that $\min u(x) = \min u_j(x(I_j)) = 0$, $\max u(x) = u(x^0) = 1$ and $\max u_j(x(I_j)) = u_j(x^0(I_j)) = M_j$ with $\sum M_j = 1$, and if u_j is approximated by

$$v_j(x(I_j)) = \prod_{i \in I_j} u_j(x_i, x_k{}^0 \text{ for } k \in I_j \backslash \{i\})/M_j^{n_j - 1},$$

then Theorem 5 above plus Theorem 5 in [2] give $D(v, u) \leqslant \sum_j D(v_j, u_j) \leqslant \sum (M_j \colon n_j > 1\}$.

Instead of additivity over subsets, it might be true (Fishburn and Keeney, [3]) that u is multiplicative over subsets. A basic multiplicative form for the partition $\{I_1,..., I_N\}$ of $\{1,..., n\}$ with $|I_j| = n_j$ and fixed point x^0 with $u(x^0) = 0$ is

$$Ku(x) + 1 = \prod_{j=1}^{N} [Ku_j(x(I_j)) + 1] \qquad \text{for all} \quad x \in T,$$

where $K \neq 0$ and $u_j(x(I_j)) = u(x_i \text{ on } I_j,\ x_i^0 \text{ on } \{1,..., n\}\backslash I_j)$. The positive affine transformation $Ku(x) + 1$ for $K > 0$ or $-Ku(x) - 1$ for $K < 0$ puts this into the form

$$u(x) = \prod_{j=1}^{N} u_j(x(I_j))/u_0^{N-1} \qquad \text{for all} \quad x \in T, \tag{10}$$

where $u_0 \in \{-1, 1\}$. If $n_j = 1$ with $I_j = \{k\}$ then $u_j(x(I_j)) = u_0(x_k)$. If $n_j \geqslant 2$ then u_j in (10) could be approximated by either an additive or multiplicative form over the $i \in I_j$. If (10) holds with $u_0 \in \{-1, 1\}$, and v_j approximates u_j with $v(x) = \prod v_j(x(I_j))/u_0^{N-1}$, then

$$D(v, u) = \max_{x \in T} \left| \prod_{j=1}^{N} u_j(x(I_j)) - \prod_{j=1}^{N} v_j(x(I_j)) \right|,$$

which can be used as a basis for further analysis.

As in Theorem 2 for the additive approximation, refinements can be made in the approach of Theorem 5 when the range of $u_0(x_i)$ is taken into consideration. To illustrate, suppose that u is scaled so that $M = \max u(x)$ and $m = \min u(x) \geqslant 0$, and let $m_i(x^0) = \min u_0(x_i)$ and $M_i(x^0) = \max u_0(x_i)$. Then, when u is not monotonic in more than $n - 2$ variables, a worst-case analysis says that

$$D(v, u) \leqslant \max \left\{ \prod_{i=1}^{n} M_i(x^0)/u(x^0)^{n-1} - m,\ M - \prod_{i=1}^{n} m_i(x^0)/u(x^0)^{n-1} \right\},$$

and similar though more complex expressions apply to the other monotonicity cases. As in the proof of Theorem 5, an effort could be made to choose x^0 to balance or equalize the terms on the right sides of these expressions. However, unlike when (2) is used, we know that the relative accuracy of v under (7), i.e. $D(v, u)/[M - m]$, depends on the choice of M and m as well as on the choice of x^0. Hence, when (7) is used, it is essential to consider the effects of scaling in addition to the choice of x^0.

4. Other Simple Approximations

The approximations in the two preceding sections are based on one conditional utility function for each variable. In this section we shall briefly examine three other approximations among the vast number that could be considered. The first of these is based on two fixed points in T and uses two conditional utility functions for each variable. The second focuses on one variable as the key aspect of the situation and uses 2^{n-1} conditional utility functions for this variable. Each of these functions corresponds to a vertex of the other $n - 1$ variables. The third and simplest approximation dispenses with conditional utility functions altogether. It uses only the values of u at the 2^n vertices of T and approximates u at other points by multilinear interpolation.

A Bilateral Approximation

The first and most complex approximation that we examine in this section corresponds to Fishburn's bilateral independence form [1], which is based on two conditional utility functions for each attribute. The approximation uses two fixed points, x^0 and x^1. Letting $u_k(x_i) = u(x_1{}^k,..., x_{i-1}^k, x_i, x_{i+1}^k,..., x_n{}^k)$ for $k = 0, 1$, the bilateral approximation is given by

$$v(x) = \sum_{i=1}^{n} u_0(x_i) - (n-1)\, u(x^0)$$
$$+ \sum \left\{ c(i_1,..., i_s) \prod_{j=1}^{s} f_{i_j}(x_{i_j}) : s \geqslant 2 \text{ and } 1 \leqslant i_1 < \cdots < i_s \leqslant n \right\} \quad (11)$$

where

$$c(i_1,..., i_s) = \sum \{(-1)^{s+\Sigma\beta_i}\, u(x_1^{\beta_1},..., x_n^{\beta_n}):$$
$$\beta_i \in \{0, 1\} \text{ for each } i \in \{i_1,..., i_s\} \text{ and } \beta_i = 0 \text{ otherwise}\},$$

$$f_i(x_i) = \frac{u_1(x_i) - u_0(x_i) + u(x^0) - u_1(x_i{}^0)}{u(x^1) + u(x^0) - u_1(x_i{}^0) - u_0(x_i{}^1)}$$

provided that the denominator of f_i does not vanish. If $u(x^1) + u(x^0) = u_1(x_i{}^0) + u_0(x_i{}^1)$ for one or more i, then (11) can be simplified as described in [1]. If $x^1 > x^0$ and u is conservative then the denominator of each f_i is nonzero.

THEOREM 6. *Suppose v is given by* (11) *with each f_i well defined. Then v is affine preserving and $v(x) = u(x)$ if either $x_i = x_i{}^0$ for at least $n - 1$ variables or $x_i = x_i{}^1$ for at least $n - 1$ variables.*

Proof. Since the transformation $au + b$, $a > 0$, sends $c(i_1,..., i_s)$ into $ac(i_1,..., i_s)$ and has no effect on f_i, it follows from (11) that v is affine preserving. If $x_i = x_i{}^0$ for all $i > 1$ then, since $f_i(x_i{}^0) = 0$ for all $i > 1$ and since each f_i product in (11) involves at least two variables, $v(x_1, x_2{}^0,..., x_n{}^0) = u_0(x_1)$. Hence, in general, $v(x) = u(x)$ when $x_i = x_i{}^0$ for at least $n - 1$

variables. On the other hand, if $x_i = x_i^1$ for all $i > 1$, then $f_i(x_i^1) = 1$ for all $i > 1$ and it can be shown without undue difficulty that (11) reduces to

$$\begin{aligned} v(x_1, x_2^1, ..., x_n^1) &= u_0(x_1) + \sum_{i>1} u_0(x_i^1) - (n-1)\, u(x^0) \\ &\quad + (n-2)\, u(x^0) - \sum_{i>1} u_0(x_i^1) + u_1(x_1^0) \\ &\quad + f_1(x_1)[u(x^1) + u(x^0) - u_1(x_1^0) - u_0(x_1^1)] \\ &= u_1(x_1). \end{aligned}$$

Therefore $v(x) = u(x)$ when $x_i = x_i^1$ for all $i > 1$, and in general $v(x) = u(x)$ when $x_i = x_i^1$ for at least $n - 1$ variables. Q.E.D.

Approximation (11) is a natural generalization of the simple additive-multiplicative form (14) in [2] and, as in the previous $n = 2$ case, a general analysis of $D(v, u)$ for (11) appears quite difficult. However, the picture simplifies greatly if u is conservative and x^0 and x^1 are fixed at the extremes of T. Then, as shown by Theorems 6 and 7, both u and v are conservative and they are equal if either at least $n - 1$ $x_i = 0$ or at least $n - 1$ $x_i = 1$.

THEOREM 7. *If u is conservative and v is specified by* (11) *with $x^0 = (0,..., 0)$ and $x^1 = (1,..., 1)$, then v is conservative.*

Proof. Let u be conservative with v given by (11) with $x^0 = (0,..., 0)$ and $x^1 = (1,..., 1)$. For definiteness we work with the first two variables. Given $x_1 > y_1$ and $x_2 > y_2$, our main task will be to show that $v(x_1, y_2, x_3, ..., x_n) - v(y_1, y_2, x_3, ..., x_n) > v(x_1, x_2, x_3, ..., x_n) - v(y_1, x_2, x_3, ..., x_n)$. This is true if and only if

$$[f_1(x_1) - f_1(y_1)][f_2(y_2) - f_2(x_2)] \left[c(1, 2) + \sum \left\{c(1, 2, i_1, ..., i_s) \prod_{j=1}^{s} f_{i_j}(x_{i_j}): s \geqslant 1 \text{ and } 3 \leqslant i_1 < \cdots i_s \leqslant n\right\}\right] > 0.$$

It is easily seen that conservatism of u implies that $f_1(x_1) - f_1(y_1) > 0$ and $f_2(y_2) - f_2(x_2) < 0$. The preceding inequality will therefore be valid if the total c term is negative. If $n = 2$ then this term is simply $c(1, 2)$, which is negative by conservatism of u. Suppose then that $n \geqslant 3$. Let h_i and d_i be respectively the numerator and denominator of $f_i(x_i)$ as defined after (11), and let $e(\gamma) = u(0, 0, \gamma) - u(1, 0, \gamma) - u(0, 1, \gamma) + u(1, 1, \gamma)$ for each $\gamma \in \{0, 1\}^{n-2}$. It then follows that

$$c(1, 2) + \sum \left\{ c(1, 2, i_1, ..., i_s) \prod_{j=1}^{s} f_{i_j}(x_{i_j}) \colon s \geqslant 1 \text{ and } 3 \leqslant i_1 < \cdots < i_s \leqslant n \right\}$$

$$= \prod_{i=3}^{n} d_i^{-1} \sum_{\gamma \in \{0,1\}^{n-2}} e(\gamma) \prod_{\{i : \gamma_i = 1\}} h_i \prod_{\{i : \gamma_i = 0\}} (d_i - h_i).$$

By conservatism of u, $d_i < 0$, $h_i < 0$, $d_i - h_i < 0$ for $i = 3, ..., n$ and $e(\gamma) < 0$ for all $\gamma \in \{0, 1\}^{n-2}$. Hence the preceding expression, or the total c term, is negative. Therefore $v(x_1, y_2, ...) - v(y_1, y_2, ...) > v(x_1, x_2, ...) - v(y_1, x_2, ...)$. Moreover, by taking the variables in sequence, $v(x_1, x_2, x_3, ... x_n) - v(y_1, x_2, x_3, ..., x_n) \geqslant v(x_1, 1, x_3, ..., x_n) - v(y_1, 1, x_3, ..., x_n) \geqslant v(x_1, 1, 1, x_4, ..., x_n) - v(y_1, 1, 1, x_4, ..., x_n) \geqslant \cdots \geqslant v(x_1, 1, ..., 1) - v(y_1, 1, ..., 1)$ when $x_1 > y_1$. By Theorem 6, $v(x_1, 1, ..., 1) - v(y_1, 1, ..., 1) = u(x_1, 1, ..., 1) - u(y_1, 1, ..., 1)$, which is positive when u is conservative and $x_1 > y_1$. Therefore v increases in its first variable when u is conservative and, by analogy, v increases in each variable when u is conservative. It then follows that v is conservative. Q.E.D.

An Approximation with One Key Variable

In many multiattribute situations one of the n attributes will be more important than the others. We now consider an approximation that seems well suited to this situation, especially when u is monotonic in its variables. The approximation is based on convex combinations of 2^{n-1} conditional utility functions of the key variable, say x_1. Each conditional function has the form $u(x_1, \lambda_2, ..., \lambda_n)$ where $\lambda = (\lambda_2, ..., \lambda_n)$ is a corner point of the other $n - 1$ variables with $\lambda_i \in \{0, 1\}$ for each i. The explicit form of the approximation is

$$v(x) = \sum_{\lambda \in \{0,1\}^{n-1}} \left[\prod_{i=2}^{n} x_i^{\lambda_i} (1 - x_i)^{1-\lambda_i} \right] u(x_1, \lambda_2, ..., \lambda_n) \qquad \text{for all} \quad x \in T \quad (12)$$

where, in the product, $0^0 = 1$. Characteristics of (12) are given in the following theorem.

THEOREM 8. *Suppose v is given by* (12). *Then v is affine preserving, monotonicity preserving, conservatism preserving, and $v(x) = u(x)$ whenever $(x_2, ..., x_n) \in \{0, 1\}^{n-1}$. Moreover, if $i > 1$ then v is a linear function of x_i when all x_j for $j \neq i$ are fixed. In addition, $D(v, u) \leqslant \max u(x) - \min u(x)$, and if u is monotone increasing in all variables then $D(v, u) \leqslant \max_{x_1} [u(x_1, 1, ..., 1) - u(x_1, 0, ..., 0)]$.*

Proof. Monotonicity preservation for x_1 is clear from (12). For $i > 1$ let $i = 2$ for definiteness. Then

$$v(x_1, x_2, x_3, \ldots, x_n)$$
$$= x_2 \sum_{\mu \in \{0,1\}^{n-2}} \left[\prod_3^n x_i^{\mu_i}(1 - x_i)^{1-\mu_i}\right] [u(x_1, 1, \mu_3, \ldots, \mu_n)$$
$$- u(x_1, 0, \mu_3, \ldots, \mu_n)] + \sum_\mu \left[\prod_3^n x_i^{\mu_i}(1 - x_i)^{1-\mu_i}\right] u(x_1, 0, \mu_3, \ldots, \mu_n),$$

which shows that v is linear in x_2 when the other x_i are fixed, and that v preserves monotonicity in x_2. Since other aspects of the theorem are obvious except for conservatism preservation, we conclude with a proof of this aspect. Assume that u is conservative. To show that v too is conservative it will suffice to consider x_1 versus x_2 and x_2 versus x_3. Suppose first that $x_1 > y_1$ and $x_2 > y_2$. By the preceding equation,

$$v(y_1, x_2, x_3, \ldots, x_n) - v(y_1, y_2, x_3, \ldots, x_n)$$
$$= (x_2 - y_2) \sum_\mu \left(\prod_3^n\right) [u(y_1, 1, \mu) - u(y_1, 0, \mu)].$$

This remains valid when y_1 is replaced by x_1 throughout. Since $x_2 > y_2$ and since $u(y_1, 1, \mu) - u(y_1, 0, \mu) > u(x_1, 1, \mu) - u(x_1, 0, \mu)$ by conservatism of u, $v(y_1, x_2, x_3, \ldots) - v(y_1, y_2, x_3, \ldots) > v(x_1, x_2, x_3, \ldots) - v(x_1, y_2, x_3, \ldots)$, which says that v is conservative in x_1 and x_2. For x_2 versus x_3 suppose that $x_2 > y_2$ and $x_3 > y_3$. By a similar procedure to that just used it follows that $v(x_1, y_2, x_3, x_4, \ldots) - v(x_1, y_2, y_3, x_4, \ldots) > v(x_1, x_2, x_3, x_4, \ldots) - v(x_1, x_2, y_3, x_4, \ldots)$, and hence that v is conservative in x_2 and x_3, if and only if

$$\sum_{\mu \in \{0,1\}^{n-3}} \left[\prod_4^n x_i^{\mu_i}(1 - x_i)^{1-\mu_i}\right] [u(x_1, 1, 0, \mu)$$
$$+ u(x_1, 0, 1, \mu) - u(x_1, 0, 0, \mu) - u(x_1, 1, 1, \mu)] > 0.$$

This is true by the conservatism of u. Q.E.D.

A Multilinear Approximation

We conclude with a simplification of the preceding approximation that is based solely on multilinear interpolation of the u values at the 2^n vertices of T. With $\lambda = (\lambda_1, \ldots, \lambda_n)$, the multilinear approximation is

$$v(x) = \sum_{\lambda \in \{0,1\}^n} \left[\prod_{i=1}^n x_i^{\lambda_i}(1 - x_i)^{1-\lambda_i}\right] u(\lambda) \qquad \text{for all} \quad x \in T. \tag{13}$$

This is the only approximation in the paper that does not require estimation of any conditional utility functions. Although it is quite simple it may serve well in some cases. The following theorem summarizes aspects of (13). Its proof is similar to the preceding proof and will be omitted.

THEOREM 9. *Suppose v is given by* (13). *Then v is affine preserving, monotonicity preserving, conservatism preserving, $v(\lambda) = u(\lambda)$ for all $\lambda \in \{0, 1\}^n$, v is linear in each x_i, and $D(v, u) \leqslant \max u(x) - \min u(x)$.*

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